The Weyl–Heisenberg group on the noncommutative two-torus: a zoo of representations

2007 J. Phys. A: Math. Theor. 40 12415

(http://iopscience.iop.org/1751-8121/40/41/011)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12415–12438 doi:10.1088/1751-8113/40/41/011

The Weyl–Heisenberg group on the noncommutative two-torus: a zoo of representations

Jan Govaerts[1,2,3,4] **and Frederik G Scholtz**[1]

[1] Institute of Theoretical Physics, Department of Physics, University of Stellenbosch, Stellenbosch 7600, Republic of South Africa
[2] UNESCO International Chair in Mathematical Physics and Applications (ICMPA), University of Abomey-Calavi, 072 BP 50, Cotonou, Republic of Benin

E-mail: Jan.Govaerts@fynu.ucl.ac.be and FGS@sun.ac.za

Received 25 June 2007
Published 25 September 2007
Online at stacks.iop.org/JPhysA/40/12415

Abstract
In order to assess possible observable effects of noncommutativity in deformations of quantum mechanics, all irreducible representations of the noncommutative Heisenberg algebra and Weyl–Heisenberg group on the two-torus are constructed. This analysis extends the well-known situation for the noncommutative torus based on the algebra of the noncommuting position operators only. When considering the dynamics of a free particle for any of the identified representations, no observable effect of noncommutativity is implied.

PACS numbers: 02.20.−a, 02.40.Gh, 03.65.Fd

1. Introduction

1.1. Motivation

The idea that space and spacetime coordinates may in fact be noncommutative goes as far back as the early days of quantum mechanics [1]. In recent years, however, it has witnessed greatly renewed interest since the issue has arisen again within attempts aiming towards a theory for quantum gravity, whether in the M-theory or loop quantum gravity contexts or more generally deformations of quantum mechanics at the smallest distance scales. Quantum field theory on noncommutative spacetimes has now grown into a research field of its own (see, e.g. [2] and references therein). In the simpler context of mechanical systems, so-called noncommutative quantum mechanics considers deformations of the ordinary Heisenberg algebra of Hermitian

[3] Fellow of the Stellenbosch Institute for Advanced Study (STIAS), Stellenbosch, Republic of South Africa.
[4] On sabbatical leave from the Center for Particle Physics and Phenomenology (CP3), Institut de Physique Nucléaire, Université catholique de Louvain (UCL), 2 Chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium.

1751-8113/07/4112415+24$30.00

operators, $\hat{x}^i$ and $\hat{p}_i$ $(i = 1, 2, \ldots, d)$, with for instance in the simplest case a nonvanishing constant space–space commutator,

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}\mathbb{I}, \qquad [\hat{x}^i, \hat{p}_j] = \mathrm{i}\hbar\delta^i_j\mathbb{I}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad i, j = 1, 2, \ldots, \tag{1}$$

the antisymmetric constants $\theta^{ij} = -\theta^{ji}$ thus parameterizing such deformations[5].

It certainly is a legitimate question to identify possible observable consequences of such noncommutative deformations of quantum mechanics, with deviations from the ordinary situation expected to become apparent at the distance scales set by the parameters θ^{ij}. However, when the operators $\hat{x}^i$ and $\hat{p}_i$ are thought of as the Cartesian coordinates spanning an Euclidean phase space, the representation theory of the noncommutative Heisenberg (NC-H) algebra (1) is not different from that of the ordinary Heisenberg algebra with $\theta^{ij} = 0$ for which, according to the Stone–von Neumann theorem, there exists a unique representation (up to unitary transformations). Indeed, by an appropriate linear change of basis in $\hat{x}^i$, the matrix θ^{ij} may be 2×2-block diagonalized. Restricted to any such two-dimensional subspace now with $i, j = 1, 2$, the NC-H algebra reduces to

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta\epsilon^{ij}\mathbb{I}, \qquad [\hat{x}^i, \hat{p}_j] = \mathrm{i}\hbar\delta^i_j\mathbb{I}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad i, j = 1, 2, \tag{2}$$

where, without loss of generality, one assumes $\theta > 0$ while $\epsilon^{ij} = \epsilon_{ij}$ is the antisymmetric symbol with $\epsilon^{12} = +1 = \epsilon_{12}$. Considering then the operators defined by the following linear combinations, corresponding to a Darboux transformation, which brings the commutation relations into canonical form,

$$\hat{X}^i = \hat{x}^i + \frac{\theta}{2\hbar}\epsilon^{ij}\hat{p}_j, \tag{3}$$

one recovers the ordinary Heisenberg algebra

$$[\hat{X}^i, \hat{X}^j] = 0, \qquad [\hat{X}^i, \hat{p}_j] = \mathrm{i}\hbar\delta^i_j\mathbb{I}, \qquad [\hat{p}_i, \hat{p}_j] = 0. \tag{4}$$

Since the abstract representation space of the algebra $(\hat{X}^i, \hat{p}_i)$ is unique and coincides in this construction with that of the original algebra $(\hat{x}^i, \hat{p}_i)$, indeed the quantum states of the deformed NC-H algebra (2) do not differ from those of the ordinary Heisenberg algebra. In other words, at the level solely of the 'kinematics' in an Euclidean configuration space, there are no observable differences between the commutative, $\theta = 0$, and noncommutative, $\theta \neq 0$, versions of the quantum commutation relations. A similar conclusion holds in the context of quantum field theory on noncommutative spacetime [3].

One may possibly object to the above argument on the grounds that the plane wave representation of the Heisenberg algebra does not define a genuine Hilbert space in a strict sense. Consequently, the linear transformation between operator representations could possibly suffer ambiguities related to the behaviour of states at infinity in the Euclidean plane. However, the restriction to states of Schwartz class is best achieved by considering the Fock algebra generators

$$\begin{aligned} b &= \frac{1}{\sqrt{2\theta}}[\hat{x}^1 + \mathrm{i}\hat{x}^2], & b^\dagger &= \frac{1}{\sqrt{2\theta}}[\hat{x}^1 - \mathrm{i}\hat{x}^2], \\ a &= b^\dagger + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{\theta}{2}}\hat{p}_-, & a^\dagger &= b - \frac{\mathrm{i}}{\hbar}\sqrt{\frac{\theta}{2}}\hat{p}_+, \end{aligned} \tag{5}$$

where $\hat{p}_\pm = \hat{p}_1 \pm \mathrm{i}\hat{p}_2$, such that the only nonvanishing commutators are

$$[b, b^\dagger] = \mathbb{I}, \qquad [a, a^\dagger] = \mathbb{I}. \tag{6}$$

5 The momentum–momentum commutator may be deformed in a likewise manner but an appropriate change of variables brings the algebra into the form (1), except for one singular choice of deformation parameters which shall not be addressed here.

Working then in the Hilbert space obtained as the closure of the separable complex vector space spanned by the Fock (and the coherent) states built out of these two commuting Fock algebras, one obtains wavefunction representations of Schwartz class of the NC-H algebra (2). It is straightforward to establish that these representations are isomorphic to the unique ordinary representation of the commutative Heisenberg algebra with $\theta = 0$ by identifying the appropriate changes of bases.

It thus follows that when configuration space is Euclidean any possible observable effect of noncommutativity must result from the dynamics, namely the specification of a Hamiltonian operator and interactions. However, in the case of a free noncommutative particle with the ordinary nonrelativistic Hamiltonian

$$\hat{H} = \frac{1}{2\mu}\delta^{ij}\hat{p}_i\hat{p}_j, \tag{7}$$

which commutes with the commuting operators $\hat{p}_i$ considered to define the generators of translations in (the eigenspectrum of) the configuration space coordinate operators $\hat{x}^i$, the energy spectrum and hence the dynamics itself clearly remains independent of the noncommutativity parameters θ^{ij} since the $\hat{p}_i$ eigenspectrum coincides with that of the commutative Heisenberg algebra. In other words, in the case of the Euclidean configuration space the manifestation of any observable effects related to noncommutativity is possible at best only in the presence of interactions (in any case, besides the physical constant $\hbar$, an extra area scale is required to combine with the noncommutativity parameter θ to construct physical observables function of θ). Obviously this is not a welcome feature since it may be difficult to disentangle effects of interactions from those of noncommutativity. Indeed, such effects may even be physically equivalent in an effective sense. There are known instances in which interactions in a given energy range within the commutative setting may be given an equivalent description in terms of noncommuting configuration space variables in the absence of any interactions safe from the coupling to an applied magnetic field [4, 5].

As an alternative one may consider configuration spaces of a topology or geometry different from those of the Euclidean space. Confining even the free particle to some potential well in effect introduces interactions through boundary conditions at the well. In the presence of coordinate noncommutativity, the specification of such boundary conditions, namely associated with a compact space with boundaries, is not straightforward and requires a dedicated formulation to be addressed elsewhere. Another form of confinement to a finite volume is through compactification of configuration space, leading to a finite area A. One might then expect that physical observables may acquire correction factors, which are functions of the ratio θ/A, while the leading order will coincide with the commutative case. The simplest choice for such a compactification is that of a torus topology. The present work addresses the dynamics of the free particle on the noncommutative two-torus associated with the noncommutative Heisenberg algebra (2). We shall proceed by first constructing all possible representations of the NC-H algebra for such a geometry, and then consider the possible dynamics of a free particle.

The rationale for the construction of representations of algebra (2) on the noncommutative two-torus (NC-2T) is as follows. Any such torus of given geometry may be seen as the quotient of the Euclidean plane by some Abelian lattice group. In terms of the NC-H algebra (2), this lattice group is realized as a specific discrete subgroup of the exponentiated noncommutative Weyl–Heisenberg (NC-WH) group of which the generators are $\mathbb{I}$, $\hat{x}^i$ and $\hat{p}_i$ $(i = 1, 2)$. Even though the coordinate operators $\hat{x}^i$ do not commute when $\theta \neq 0$, what is required is only that the group composition law for the lattice subgroup be Abelian, namely additive in the lattice vectors. This requirement should entail a quantized cocycle condition in the noncommutative case. Having thereby constructed the appropriate lattice group associated with a given NC-2T geometry, it remains to identify within the unique representation space of the NC-H algebra (2) on the plane those states that are left invariant under the action of the lattice group, as well as those elements of the full NC-WH group generated by (2) which commute with the lattice subgroup of the NC-WH group, namely the normalizer of the lattice subgroup within the NC-WH group. By construction, the elements of the latter normalizer then map invariant states into one another in a single-valued manner on the NC-2T. In other words, the set of invariant states defines a closed representation space for the NC-WH subgroup which commutes with the lattice group characterizing the noncommutative two-torus. The set of such possible representations associated with a given torus geometry then provides the realm from which to choose a realization of the noncommutative particle's motion.

In the present case the choice of dynamics, namely of the Hamiltonian operator, should reflect the free character of particle's motion on the noncommutative two-torus. This is best achieved in an invariant manner, by requiring, as in the ordinary commutative case, that the Hamiltonian commutes with the generators of space translations. We take this requirement to define what is meant by a free particle, whether in the commutative or the noncommutative context. Hence the Hamiltonian will be chosen to be quadratic in the operators which commute with the translation generators. Since the lattice group is certainly to be constructed in terms of the translation operators, the action of such a Hamiltonian operator preserves the invariant character of quantum states, hence it acts within any of the possible representations of the NC-WH group on the NC-2T.

1.2. Methodology

The construction thus relies entirely, on the one hand, on the choice of the lattice vectors specifying the geometry of the two-torus, and on the other hand, on the specification of the translation operators. The lattice vectors are to be denoted e_a^i $(a = 1, 2; i = 1, 2)$ with the following identifications in the spectrum of $\hat{x}^i$ eigenvalues defining the two-torus[6]:

$$x^i \sim x^i + n^a e_a^i, \qquad n^a \in \mathbb{Z}. \tag{8}$$

Denoting by $\hat{T}_i$ the translation generators in configuration space, lattice group elements must be of the form

$$U(n^a) = C(n^a)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar} n^a e_a^i \hat{T}_i}, \tag{9}$$

where $C(n^a)$ are cocycle factors to be chosen such that the Abelian group composition law of the lattice, additive in the lattice vectors $n^a e_a^i$ and $\ell^a e_a^i$, be obeyed

$$U(n^a)U(\ell^a) = U(n^a + \ell^a), \qquad n^a, \ell^a \in \mathbb{Z}, \tag{10}$$

irrespective of whether the operators $\hat{T}_i$ commute with one another or not. The choice of translation operators $\hat{T}_i$ must be such that their adjoint action on the coordinate operators $\hat{x}^i$ induces the appropriate lattice shift,

$$U^\dagger(n^a)\hat{x}^i U(n^a) = \hat{x}^i + n^a e_a^i \mathbb{I}, \tag{11}$$

a condition which requires the property

$$[\hat{x}^i, \hat{T}_j] = \mathrm{i}\hbar\delta_j^i \mathbb{I}. \tag{12}$$

In the ordinary commutative context, the translation generators are taken to coincide with the conjugate momentum operators, $\hat{T}_i = \hat{p}_i$, in which case these operators commute and are left invariant by the lattice group spanned by $U(n^a)$. However, in the present context, there is *a priori* nothing to prevent us from considering more general linear combinations of

[6] See the appendix for a compendium of useful properties of these lattice vectors and their dual vectors $\tilde{e}_i^a$.

the basic operators $\hat{x}^i$ and $\hat{p}_i$ such that the conditions (12) are met. In the noncommutative case, the coordinate operators $\hat{x}^i$ certainly also effect translations in configuration space, while the commuting momentum operators $\hat{p}_i$ may in fact then result from linear combinations of $\hat{x}^i$ with originally noncommuting momentum operators. Certainly in the presence of noncommutativity the distinction between the configuration and momentum spaces is less clear-cut than in the commutative case, and while one translates in configuration space translations in momentum space may also be induced on a scale set by $\hbar/\sqrt{\theta}$. From this point of view, we take here the definition of the torus geometry to be given by relation (11) irrespective of the transformation properties of the momentum operators under the lattice group operators $U(n^a)$. Note that such a characterization of the lattice group and the torus geometry allows even in the commutative case a more general choice for translation operators than simply the momenta $\hat{p}_i$ as is usually done. Since the possibility offers itself, it certainly is worth exploring its consequences and possible physical relevance.

Once a choice of translation generators $\hat{T}_i$ has been made in accordance with (12), as well as lattice group elements $U(n^a)$ in (9) with cocycle factors $C(n^a)$ in compliance with the Abelian group composition law (10), it is possible to identify the subspace of quantum states of the unique representation space for the NC-H algebra (2) on the noncommutative plane which are invariant under the lattice group, namely, it is the quotient of the original representation space by the lattice group spanned by $U(n^a)$. This invariant subspace may also be determined by considering the (non-normalizable) projector (density)

$$\mathbb{P} = \sum_{n^a \in \mathbb{Z}} U(n^a) \tag{13}$$

applied on the original representation space.

What then remains to be done is to identify the subgroup of the NC-WH group, generated by the NC-H algebra (2), for which the action on these states closes in a manner consistent with the lattice group action. More specifically, the general unitary operators representing elements of the NC-WH group generated by (2) are parameterized according to

$$W(x^i, p_i; \varphi) = \exp\left[\mathrm{i}\varphi\mathbb{I} + \frac{\mathrm{i}}{\hbar} p_i \hat{X}^i - \frac{\mathrm{i}}{\hbar} X^i \hat{p}_i\right] = \exp\left[\mathrm{i}\varphi\mathbb{I} + \frac{\mathrm{i}}{\hbar} p_i \hat{x}^i - \frac{\mathrm{i}}{\hbar}\left(x^i + \frac{\theta}{\hbar}\epsilon^{ij} p_j\right)\hat{p}_i\right]. \tag{14}$$

Here,

$$X^i = x^i + \frac{\theta}{2\hbar}\epsilon^{ij} p_j, \tag{15}$$

with x^i, p_i and φ (defined modulo 2π) the real parameters spanning the NC-WH group. The reason for this specific choice of parameterization in terms of the commuting Heisenberg algebra, associated with $(\hat{X}^i, \hat{p}_i, \mathbb{I})$, is that the adjoint action of the unitary operators $W(x^i, p_i; \varphi)$ (with $W^\dagger(x^i, p_i; \varphi) = W^{-1}(x^i, p_i; \varphi) = W(-x^i, -p_i; -\varphi)$) is then indeed such that the operators $\hat{x}^i$ and $\hat{p}_i$ are shifted by the constant parameters x^i and p_i, respectively, and subsequently also their eigenspectra[7],

$$W^\dagger(x^i, p_i; \varphi)\hat{x}^i W(x^i, p_i; \varphi) = \hat{x}^i + x^i\mathbb{I}, \qquad W^\dagger(x^i, p_i; \varphi)\hat{p}_i W(x^i, p_i; \varphi) = \hat{p}_i + p_i\mathbb{I}. \tag{16}$$

The lattice group elements $U(n^a)$ are a particular subclass of these operators with parameters $(x^i, p_i; \varphi)$ given by specific functions of $n^a \in \mathbb{Z}$. We thus have

[7] The remaining generator $\mathbb{I}$ of the NC-H algebra is of course invariant under this adjoint action.

$$U^\dagger(n^a)\hat{x}^i U(n^a) = \hat{x}^i + \Delta_n x^i \mathbb{I}, \qquad \Delta_n x^i = n^a e^i_a,$$
$$U^\dagger(n^a)\hat{p}_i U(n^a) = \hat{p}_i + \Delta_n p_i \mathbb{I}, \qquad \Delta_n p_i = n^a \Delta_a p_i, \tag{17}$$

where $\Delta_a p_i$ depend on the specific choice of translation generators $\hat{T}_i$.

Requiring now consistency between the action of the NC-WH group elements $W(x^i, p_i; \varphi)$ and the lattice group elements $U(n^a)$ will restrict the ranges for the NC-WH group parameters $(x^i, p_i; \varphi)$ in such a way that the associated subclass still closes into a subgroup of the original NC-WH group, namely the noncommutative two-torus Weyl–Heisenberg (NC-2T-WH) group, and commutes with the lattice group. The action of the NC-2T-WH group then closes on the subspace of invariant states. The latter condition corresponds to the requirement that, for all $n^a \in \mathbb{Z}$,

$$U(n^a)W(x^i, p_i; \varphi) = W(x^i, p_i; \varphi)U(n^a), \tag{18}$$

leading to restrictions on the NC-WH group parameters $(x^i, p_i; \varphi)$.

Furthermore, any such restricted NC-WH group element $W(x^i, p_i; \varphi)$ acting on an invariant state produces another invariant state which must be single-valued in lattice shifts of the parameters (x^i, p_i). Due to the possible nontrivial cocycle factor $C(n^a)$ in $U(n^a)$, as well as other phase factors arising from combining the product $U(n^a)W(x^i, p_i; \varphi)$ into a new element of the form $W(x^i + \Delta_n x^i, p_i + \Delta_n p_i; \varphi')$, this condition of single-valuedness requires a specific dependence $\varphi(x^i, p_i)$ for the phase parameter φ such that one meets a second restriction of the form

$$U(n^a)W(x^i, p_i; \varphi(x^i, p_i)) = W(x^i(n), p_i(n); \varphi(x^i(n), p_i(n))) = W(x^i, p_i; \varphi)U(n^a), \tag{19}$$

for all $n^a \in \mathbb{Z}$. Here, $x^i(n) = x^i + \Delta_n x^i$ and $p_i(n) = p_i + \Delta_n p_i$.

Provided the two conditions (18) and (19) are met, any invariant state, $U(n^a)|\psi\rangle = |\psi\rangle$, is then mapped into an invariant state,

$$U(n^a)W(x^i, p_i; \varphi)|\psi\rangle = W(x^i, p_i; \varphi)U(n^a)|\psi\rangle = W(x^i, p_i; \varphi)|\psi\rangle, \tag{20}$$

while any of its NC-2T-WH images is single-valued in any lattice shift of the group parameters,

$$W(x^i(n), p_i(n); \varphi(x^i(n), p_i(n)))|\psi\rangle = W(x^i, p_i; \varphi(x^i, p_i))U(n^a)|\psi\rangle$$
$$= W(x^i, p_i; \varphi(x^i, p_i))|\psi\rangle. \tag{21}$$

Note that in actual fact none of the above considerations requires the specification of an inner product on the representation space of the NC-H algebra (2) on the noncommutative plane. It is true that such a structure is required to ensure the hermiticity and unitarity properties mentioned throughout the above discussion, but, as a matter of fact, one is free to introduce a different, or new inner product on the final representation space obtained as the quotient by the lattice group, and still fulfil the necessary properties of hermiticity and unitarity. This freedom in a (re)definition of the inner product often allows for normalizable invariant states when the invariant representation space is discrete or even of finite dimension, in contradistinction to the situation in the original representation space.

The above general description outlines the approach which is to be developed hereafter. For the purpose of illustration and later comparison with the noncommutative situation, these considerations are applied in the following section to the general d-dimensional torus in the case of the ordinary commuting Heisenberg algebra (with $\theta^{ij} = 0$ in (1)). In section 3, the same considerations are applied to the ordinary noncommutative configuration space subalgebra

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta\epsilon^{ij}\mathbb{I}, \qquad \theta > 0, \qquad i, j = 1, 2, \tag{22}$$

which does not yet include the momentum operators $\hat{p}_i$. The representation theory of this structure on the noncommutative two-torus is of course well known [6]. It is rederived here for the purpose of establishing the consistency of the above construction, and more importantly to show how, by extending the algebra to include the commuting momentum operators $\hat{p}_i$, the representation theory on the two-torus becomes drastically different. Section 4 finally addresses the situation of interest associated with algebra (2), and establishes the quantized cocycle condition in terms of an integer quantity $k_0 \in \mathbb{Z}$. The latter quantization condition possesses two distinguished solutions associated with $k_0 = 0$, considered in section 5, and a generic branch associated with $k_0 \neq 0$, discussed in section 6. The results detailed in these three sections thus provide the representation theory of the noncommutative two-torus Weyl–Heisenberg group. Finally, section 7 identifies the free Hamiltonian based on the considerations mentioned previously, and determines the energy spectrum of the free noncommutative particle on the two-torus for each of the established representations. The discussion ends with some conclusions. An appendix collects conventions and properties for the two-torus geometry.

2. The ordinary general torus

In the case of the ordinary commutative Heisenberg algebra on the Euclidean d-dimensional plane, the unitary Weyl–Heisenberg group elements are parameterized according to

$$W(x^i, p_i; \varphi) = \exp\left[\mathrm{i}\varphi\mathbb{I} + \frac{\mathrm{i}}{\hbar}p_i\hat{x}^i - \frac{\mathrm{i}}{\hbar}x^i\hat{p}_i\right], \tag{23}$$

where $x^i, p_i \in \mathbb{R}$ and $\varphi \in [0, 2\pi[$ (mod 2π). The group composition law is[8]

$$W\big(x_2^i, p_{2i}; \varphi_2\big)W\big(x_1^i, p_{1i}; \varphi_1\big) = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(p_{2i}x_1^i - x_2^i p_{1i})}W\big(x_2^i + x_1^i, p_{2i} + p_{1i}; \varphi_2 + \varphi_1\big), \tag{24}$$

from which the following cocycle property follows:

$$W\big(x_1^i, p_{1i}; \varphi_1\big)W\big(x_2^i, p_{2i}; \varphi_2\big) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p_{1i}x_2^i - p_{2i}x_1^i)}W\big(x_2^i, p_{2i}; \varphi_2\big)W\big(x_1^i, p_{1i}; \varphi_1\big). \tag{25}$$

This algebra and group are represented in the usual way with as bases, say, the position, $|x^i\rangle$, or momentum, $|p_i\rangle$, eigenbasis of the position, $\hat{x}^i$, and momentum, $\hat{p}_i$, operators, respectively,

$$\hat{x}^i|x^i\rangle = x^i|x^i\rangle, \qquad \hat{p}_i|p_i\rangle = p_i|p_i\rangle, \qquad x^i, p_i \in \mathbb{R}. \tag{26}$$

Even though the inner product of these bases vectors need not be specified at this stage, their relative phases may be fixed as follows:

$$|x^i\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}x^i\hat{p}_i}|x^i = 0\rangle, \qquad |p_i\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_i\hat{x}^i}|p_i = 0\rangle, \tag{27}$$

with the properties

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}x_0^i\hat{p}_i}|x^i\rangle = \big|x^i + x_0^i\big\rangle, \qquad \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_{0i}\hat{x}^i}|p_i\rangle = |p_i + p_{0i}\rangle. \tag{28}$$

As translation operators, in the present context, we make the usual choice $\hat{T}_i = \hat{p}_i$, which is a commuting set of operators. It thus proves convenient henceforth to work in the momentum eigenbasis $|p_i\rangle$.

The d-dimensional torus geometry, T_d, is characterized by the lattice vectors e_a^i $(a, i = 1, 2, \ldots, d)$, with their dual vectors $\tilde{e}_i^a$ such that $e_a^i\tilde{e}_i^b = \delta_a^b$ and $\tilde{e}_i^a e_a^j = \delta_i^j$, leading to the lattice identification $x^i \sim x^i + n^a e_a^i$ $(n^a \in \mathbb{Z})$ defining the torus. Consequently, the lattice group consists of the following elements, providing the general solution to the composition rule (10),

$$U(n^a) = \mathrm{e}^{2\mathrm{i}\pi n^a\lambda_a}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}n^a e_a^i\hat{p}_i} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}n^a e_a^i(\hat{p}_i - 2\pi\hbar\tilde{e}_i^a\lambda_a)} = W\big(n^a e_a^i, 0; 2\pi n^a\lambda_a\big), \tag{29}$$

[8] The identities $e^Ae^B = e^{A+B+[A,B]/2}$ and $e^ABe^{-A} = A + [A, B]$, valid when both A and B commute with their commutator $[A, B]$, are used throughout.

where $\lambda_a \in \mathbb{R}$, defined modulo the integers, are $U(1)$ holonomy factors labelling inequivalent representations of the Heisenberg algebra on the T_d torus (see, e.g. [7] and references therein), thus also characterizing the cocycle factors $C(n^a)$, $C(n^a) = \exp(2i\pi n^a \lambda_a)$. Note that lattice shift transformations of the Weyl–Heisenberg group parameters $(x^i, p_i; \varphi)$ are then

$$\Delta_n x^i = n^a e^i_a, \qquad \Delta_n p_i = 0. \tag{30}$$

It is also obvious that the subspace of invariant states is spanned by all the momentum eigenstates belonging to the following discrete set:

$$|\overline{m}_a\rangle \equiv |\overline{p}_i\rangle, \qquad \overline{p}_i = 2\pi\hbar\tilde{e}^a_i[\overline{m}_a + \lambda_a], \qquad \overline{m}_a \in \mathbb{Z}. \tag{31}$$

The same identification follows from considering the projection operator (13).

In order to determine the subgroup of Weyl–Heisenberg elements $W(x^i, p_i; \varphi)$ which commutes with the lattice group, the composition rule (24) implies that the condition (18) imposes the restriction

$$W(x^i, p_i; \varphi): \qquad p_i = 2\pi\hbar\tilde{e}^a_i m_a, \qquad m_a \in \mathbb{Z}. \tag{32}$$

Furthermore, using now (25), the second condition (19) is obeyed provided the phase parameter φ is restricted to the form

$$W(x^i, p_i; \varphi): \qquad p_i = 2\pi\hbar\tilde{e}^a_i m_a, \qquad \varphi = \pi x^i \tilde{e}^a_i\,(m_a + 2\lambda_a)\,. \tag{33}$$

Consequently, the Weyl–Heisenberg group for this torus geometry consists of all operators of the form

$$W_0(x^i, m_a) = W\big(x^i, 2\pi\hbar\tilde{e}^a_i m_a; \pi x^i \tilde{e}^a_i(m_a + 2\lambda_a)\big) = \mathrm{e}^{2i\pi\tilde{e}^a_i m_a \hat{x}^i}\,\mathrm{e}^{-\frac{i}{\hbar}x^i(\hat{p}_i - 2\pi\hbar\tilde{e}^a_i\lambda_a)}, \tag{34}$$

labelled by the parameters $x^i \in \mathbb{R}$ and $m_a \in \mathbb{Z}$. Under lattice shifts, these parameters vary according to

$$\Delta_n x^i = n^a e^i_a, \qquad \Delta_n m_a = 0. \tag{35}$$

Given the previously specified phase convention for the momentum eigenstates, the representation of the Weyl–Heisenberg group on the space of invariant states is given by

$$W_0(x^i, m_a)|\overline{m}_a\rangle = \mathrm{e}^{-2i\pi x^i\tilde{e}^a_i\overline{m}_a}|\overline{m}_a + m_a\rangle. \tag{36}$$

Since this action is single-valued under lattice shifts $\big(\Delta_n x^i = n^a e^i_a, \Delta_n m_a = 0\big)$ of the parameters (x^i, m_a), it suffices to restrict x^i to the fundamental domain of the lattice defining the torus, $x^i = u^a e^i_a, u^a \in [0, 1[$. However, all values $m_a \in \mathbb{Z}$ are required, so that the representation space spanned by all states $|\overline{m}_a\rangle$ with $\overline{m}_a \in \mathbb{Z}$ is indeed irreducible under the action of the torus Weyl–Heisenberg group.

Finally, the composition rule of this commutative torus Weyl–Heisenberg group is

$$W_0\big(x^i_2, m_{2a}\big)W_0\big(x^i_1, m_{1a}\big) = \mathrm{e}^{-2i\pi x^i_2\tilde{e}^a_i m_{1a}}W_0\big(x^i_2 + x^i_1, m_{2a} + m_{1a}\big), \tag{37}$$

from which follows the cocycle property

$$W_0\big(x^i_1, m_{1a}\big)W_0\big(x^i_2, m_{2a}\big) = \mathrm{e}^{2i\pi(x^i_2\tilde{e}^a_i m_{1a} - x^i_1\tilde{e}^a_i m_{2a})}W_0\big(x^i_2, m_{2a}\big)W_0\big(x^i_1, m_{1a}\big). \tag{38}$$

Hence, for each choice of $U(1)$ holonomy parameters $\lambda_a \in [0, 1[$ (mod $\mathbb{Z}$), one obtains an irreducible countable infinite dimensional representation of the Weyl–Heisenberg group on the d-dimensional torus, spanned by the states $|\overline{m}_a\rangle, \overline{m}_a \in \mathbb{Z}$. One may now (re)specify the inner product on that representation space, ensuring all the required hermiticity and unitarity properties of operators, with the orthonormalized choice

$$\langle\overline{m}_a|\overline{\ell}_a\rangle = \delta^{(d)}_{\overline{m},\overline{\ell}}. \tag{39}$$

That different choices of holonomy parameters $\lambda_a \in [0, 1[$ correspond to unitarily inequivalent representations may be seen, for instance, by noting that the momentum spectrum of invariant states is given as $\overline{p}_i = 2\pi\hbar\tilde{e}_i^a(\overline{m}_a + \lambda_a)$, $\overline{m}_a \in \mathbb{Z}$. All these results are well known. However, the above discussion serves the purpose of illustrating in a simple case the general methodology of this paper, while also sharing quite many aspects with parts of the analysis hereafter.

As a final remark, note that the composition rule (37) allows one to also readily identify finite or infinite discrete subgroups of the torus Weyl–Heisenberg group in terms of subsets of the parameters (x^i, m_a) which are closed under the addition rule defined by (37). The representation space spanned by $|\overline{m}_a\rangle$ may or may not become reducible under such group reductions. However, it is important to keep in mind that one is then no longer dealing with the torus Weyl–Heisenberg group, but only a subgroup of it, and possibly then even only a subalgebra of the original Heisenberg algebra spanned by $\hat{x}^i$, $\hat{p}_i$ and $\mathbb{I}$, as the case may be.

3. The ordinary noncommutative torus

Let us now turn to the noncommutative algebra (22) spanned only by the three operators $\hat{x}^i$ $(i = 1, 2)$ and $\mathbb{I}$. Given the two-torus geometry to be considered hereafter, characterized by the lattice vectors[9] e_a^i $(a, i = 1, 2)$, it is convenient to work with the 'rectified' coordinate operators

$$\hat{u}^a = \hat{x}^i\tilde{e}_i^a, \qquad \hat{x}^i = \hat{u}^a e_a^i, \tag{40}$$

such that[10]

$$[\hat{u}^a, \hat{u}^b] = \mathrm{i}\frac{\theta}{A}\epsilon^{ab}\mathbb{I}. \tag{41}$$

The elements of the non-Abelian group associated with this noncommutative algebra are parameterized as follows:

$$W(u^a; \varphi) = \mathrm{e}^{\mathrm{i}\varphi\mathbb{I} - \mathrm{i}\frac{A}{\theta}u^a\epsilon_{ab}\hat{u}^b}, \tag{42}$$

in terms of parameters $u^a \in \mathbb{R}$ and $\varphi \in [0, 2\pi[$ (mod 2π) and such that

$$W^\dagger(u^a; \varphi)\hat{u}^a W(u^a; \varphi) = \hat{u}^a + u^a\mathbb{I}. \tag{43}$$

The group composition law is

$$W\big(u_2^a; \varphi_2\big)W\big(u_1^a; \varphi_1\big) = \mathrm{e}^{-\frac{\mathrm{i}A}{2\theta}\epsilon_{ab}u_2^a u_1^b}W\big(u_2^a + u_1^a; \varphi_2 + \varphi_1\big), \tag{44}$$

from which follows the cocycle property,

$$W\big(u_1^a; \varphi_1\big)W\big(u_2^a; \varphi_2\big) = \mathrm{e}^{\frac{\mathrm{i}A}{\theta}\epsilon_{ab}u_2^a u_1^b}W\big(u_2^a; \varphi_2\big)W\big(u_1^a; \varphi_1\big). \tag{45}$$

The representation space of this algebra and group is spanned in terms of either $\hat{u}^1$ or $\hat{u}^2$ eigenstates, $|u^1\rangle_1$ or $|u^2\rangle_2$, respectively,

$$\hat{u}^1|u^1\rangle_1 = u^1|u^1\rangle_1, \qquad \hat{u}^2|u^2\rangle_2 = u^2|u^2\rangle_2. \tag{46}$$

Here again let us only specify the relative phases of these states, but not yet their inner product, through the definitions

$$|u^1\rangle_1 = \mathrm{e}^{-\frac{\mathrm{i}A}{\theta}u^1\hat{u}^2}|u^1 = 0\rangle_1, \qquad |u^2\rangle_2 = \mathrm{e}^{\frac{\mathrm{i}A}{\theta}u^2\hat{u}^1}|u^2 = 0\rangle_2, \tag{47}$$

a choice which implies the properties

$$\mathrm{e}^{-\frac{\mathrm{i}A}{\theta}u_0^1\hat{u}^2}|u^1\rangle_1 = \big|u^1 + u_0^1\big\rangle_1, \qquad \mathrm{e}^{\frac{\mathrm{i}A}{\theta}u_0^2\hat{u}^1}|u^2\rangle_2 = \big|u^2 + u_0^2\big\rangle_2. \tag{48}$$

[9] Further properties and conventions are specified in the appendix.

[10] In the present discussion, the ratio θ/A thus plays a rôle akin to that of Planck's constant $\hbar$ in the one-dimensional Heisenberg algebra $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$ given the associations $\hat{u}^1 \leftrightarrow \hat{x}$ and $\hat{u}^2 \leftrightarrow \hat{p}$.

As translation operators in the present case there is no other choice possible than $\hat{T}_i = \tilde{e}_i^a \hat{T}_a$ with $\hat{T}_a = \epsilon_{ab}\hat{u}^b$, leading to the lattice group elements

$$U(n^a) = C(n^a)\,\mathrm{e}^{-\frac{\mathrm{i}A}{\theta} n^a \epsilon_{ab}\hat{u}^b}. \tag{49}$$

The Abelian composition law condition (10) implies the following cocycle property:

$$\mathrm{e}^{-\frac{\mathrm{i}A}{2\theta}\epsilon_{ab}n^a\ell^b} C(n^a)C(\ell^a) = C(n^a + \ell^a), \tag{50}$$

for which the general solution is given by

$$C(n^a) = \mathrm{e}^{-\mathrm{i}\pi k_0 n^1 n^2}\,\mathrm{e}^{2\mathrm{i}\pi n^a\epsilon_{ab}\lambda^b}, \tag{51}$$

$k_0 \in \mathbb{N}^*$ being a positive natural number in terms of which the torus area A is quantized in units of $2\pi\theta$,

$$A = 2\pi\theta k_0, \qquad k_0 \in \mathbb{N}^*. \tag{52}$$

This labels a semi-infinite discrete series of representations, where, once again, $\lambda^a \in [0, 1[$ (modulo the integers) are $U(1)$ holonomy parameters labelling unitarily inequivalent representations of the noncommutative two-torus group for each value of k_0. Given these choices, one thus has

$$U(n^a) = \mathrm{e}^{2\mathrm{i}\pi k_0 n^2(\hat{u}^1 - \frac{\lambda^1}{k_0})}\,\mathrm{e}^{-2\mathrm{i}\pi k_0 n^1(\hat{u}^2 - \frac{\lambda^2}{k_0})} = \mathrm{e}^{-2\mathrm{i}\pi k_0 n^1(\hat{u}^2 - \frac{\lambda^2}{k_0})}\,\mathrm{e}^{2\mathrm{i}\pi k_0 n^2(\hat{u}^1 - \frac{\lambda^1}{k_0})} \tag{53}$$

with the identification

$$U(n^a) = W(n^a, 2\pi n^a\epsilon_{ab}\lambda^b - \pi k_0 n^1 n^2). \tag{54}$$

Note that under lattice shifts the group parameters u^a transform according to

$$\Delta_n u^a = n^a, \qquad \Delta_a u^b = \delta_a^b. \tag{55}$$

Invariant states may be identified in the $|u^1\rangle_1$ or $|u^2\rangle_2$ basis either by direct construction or by considering the action of the projection operator (13). In the $|u^2\rangle_2$ basis, one finds the following collection of invariant states:

$$|\overline{k}^2\rangle\rangle_2 = \sum_{\ell^2=-\infty}^{+\infty} \mathrm{e}^{-2\mathrm{i}\pi\ell^2\lambda^1}|\overline{u}^2 + \ell^2\rangle_2, \qquad \overline{u}^2 = \frac{\overline{k}^2 + \lambda^2}{k_0}, \qquad \overline{k}^2 \in \mathbb{Z}, \tag{56}$$

and likewise in the $|u^1\rangle_1$ basis,

$$|\overline{k}^1\rangle\rangle_1 = \sum_{\ell^1=-\infty}^{+\infty} \mathrm{e}^{2\mathrm{i}\pi\ell^1\lambda^2}|\overline{u}^1 + \ell^1\rangle_1, \qquad \overline{u}^1 = \frac{\overline{k}^1 + \lambda^1}{k_0}, \qquad \overline{k}^1 \in \mathbb{Z}. \tag{57}$$

However, because of the following properties, for $n^1, n^2 \in \mathbb{Z}$,

$$|\overline{k}^2 + k_0 n^2\rangle\rangle_2 = \mathrm{e}^{2\mathrm{i}\pi n^2\lambda^1}|\overline{k}^2\rangle\rangle_2, \qquad |\overline{k}^1 + k_0 n^1\rangle\rangle_1 = \mathrm{e}^{-2\mathrm{i}\pi n^1\lambda^2}|\overline{k}^1\rangle\rangle_1, \tag{58}$$

one obtains at each instance a finite k_0-dimensional space of invariant states, labelled by the integers $\overline{k}^2$ or $\overline{k}^1$ defined modulo k_0.

Given the identification (54) and the composition law (44), it is readily seen that the requirement (18) is met provided the parameters u^a labelling group transformations are such that

$$u^a = \frac{k^a}{k_0}, \qquad k^a \in \mathbb{Z}. \tag{59}$$

Under lattice shifts we thus also have

$$\Delta_n k^a = k_0 n^a, \qquad \Delta_a k^b = k_0\delta_a^b. \tag{60}$$

This equivalence relation for group elements is enforced in a consistent way by also considering the requirement (19), which is met provided the group parameter φ is also restricted as follows when $u^a = k^a/k_0$,

$$\varphi(u^a) = \pi\frac{k^1k^2}{k_0} + 2\pi\epsilon_{ab}\frac{k^a\lambda^b}{k_0}. \tag{61}$$

Consequently, the noncommutative two-torus group consists of all the operators of the form

$$W_0(k^a) = W\left(\frac{k^a}{k_0}; \pi\frac{k^1k^2}{k_0} + 2\pi\epsilon_{ab}\frac{k^a\lambda^b}{k_0}\right) = \mathrm{e}^{\mathrm{i}\pi\frac{k^1k^2}{k_0}}\,\mathrm{e}^{-2\mathrm{i}\pi k^a\epsilon_{ab}(\hat{u}^b - \frac{\lambda^b}{k_0})}, \tag{62}$$

labelled by the integers $k^a \in \mathbb{Z}$. That these integers are defined modulo k_0 follows from the action on the invariant states,

$$W_0(k^a)|\overline{k}^2\rangle\rangle_2 = \mathrm{e}^{-2\mathrm{i}\pi\frac{k^1\overline{k}^2}{k_0}}\,\mathrm{e}^{-2\mathrm{i}\pi\frac{k^2\lambda^1}{k_0}}\,|\overline{k}^2 + k^2\rangle\rangle_2, \tag{63}$$

$$W_0(k^a)|\overline{k}^1\rangle\rangle_1 = \mathrm{e}^{2\mathrm{i}\pi\frac{k^1k^2}{k_0}}\,\mathrm{e}^{2\mathrm{i}\pi\frac{k^2\overline{k}^1}{k_0}}\,\mathrm{e}^{2\mathrm{i}\pi\frac{k^1\lambda^2}{k_0}}\,|\overline{k}^1 + k^1\rangle\rangle_1, \tag{64}$$

which are indeed single-valued under lattice shifts $\Delta_n k^a = k_0 n^a$, provided the properties (58) are taken into account.

The group composition law is

$$W_0(k^a)W_0(\ell^a) = \mathrm{e}^{-\frac{2\mathrm{i}\pi}{k_0}k^1\ell^2}\,W_0(k^a + \ell^a), \tag{65}$$

leading to the cocycle property

$$W_0(\ell^a)W_0(k^a) = \mathrm{e}^{-\frac{2\mathrm{i}\pi}{k_0}\epsilon_{ab}\ell^a k^b}\,W_0(k^a)W_0(\ell^a). \tag{66}$$

In conclusion, given the quantized torus area $A = 2\pi\theta k_0$, the noncommutative two-torus group is finite dimensional, consists of k_0^2 elements and is generated from the two basic elements g_1 and g_2 given by

$$g_1 = W_0(k^1 = 1, k^2 = 0), \qquad g_2 = W_0(k^1 = 0, k^2 = 1), \tag{67}$$

which are such that

$$g_2 g_1 = \mathrm{e}^{\frac{2\mathrm{i}\pi}{k_0}}\,g_1 g_2. \tag{68}$$

The representation space of this group is k_0-dimensional, and is spanned by either the states $|\overline{k}^2\rangle\rangle_2$ or $|\overline{k}^1\rangle\rangle_1$, where $\overline{k}^a = 0, 1, 2, \ldots, k_0 - 1$ $(a = 1, 2)$. It is possible to define an inner product on this space, such that all the hermiticity and unitarity properties are obeyed, in terms of the orthonormalization conditions

$${}_2\langle\langle\overline{k}^2|\overline{\ell}^2\rangle\rangle_2 = \delta_{\overline{k}^2,\overline{\ell}^2}, \qquad {}_1\langle\langle\overline{k}^1|\overline{\ell}^1\rangle\rangle_1 = \delta_{\overline{k}^1,\overline{\ell}^1}, \tag{69}$$

as well as the overlap functions

$${}_1\langle\langle\overline{k}^1|\overline{k}^2\rangle\rangle_2 = \frac{1}{\sqrt{k_0}}\,\mathrm{e}^{\frac{2\mathrm{i}\pi}{k_0}(\overline{k}^1+\lambda^1)(\overline{k}^2+\lambda^2)}. \tag{70}$$

Except for the presence of the $U(1)$ holonomy parameters $\lambda^a \in [0, 1[$, these results are well known [6]. Still they are included here in order to show how they follow from the methodology outlined in the introduction, and to contrast them with the results for the representation theory of the full Weyl–Heisenberg group on the noncommutative two-torus.

4. The noncommutative Weyl–Heisenberg algebra on the torus

Let us now turn to the full noncommutative Heisenberg algebra (2) on the noncommutative Euclidean plane. We define the following basis of operators in terms of the lattice vectors e^i_a defining the two-torus geometry to be considered presently,

$$\hat{u}^a = \hat{x}^i \tilde{e}^a_i, \qquad \hat{v}_a = e^i_a \hat{p}_i; \qquad \hat{x}^i = \hat{u}^a e^i_a, \qquad \hat{p}_i = \tilde{e}^a_i \hat{v}_a. \tag{71}$$

The NC-H algebra then reads

$$[\hat{u}^a, \hat{u}^b] = \mathrm{i}\frac{\theta}{A}\epsilon^{ab}\mathbb{I}, \qquad [\hat{u}^a, \hat{v}_b] = \mathrm{i}\hbar\delta^a_b\mathbb{I}, \qquad [\hat{v}_a, \hat{v}_b] = 0. \tag{72}$$

Introducing also

$$\hat{U}^a = \hat{X}^i \tilde{e}^a_i = \hat{u}^a + \frac{\theta}{2A\hbar}\epsilon^{ab}\hat{v}_b, \qquad \hat{u}^a = \hat{U}^a - \frac{\theta}{2A\hbar}\epsilon^{ab}\hat{v}_b, \tag{73}$$

the algebra becomes of the ordinary commutative type,

$$[\hat{U}^a, \hat{U}^b] = 0, \qquad [\hat{U}^a, \hat{v}_b] = \mathrm{i}\hbar\delta^a_b\mathbb{I}, \qquad [\hat{v}_a, \hat{v}_b] = 0. \tag{74}$$

Hence the unique representation space is spanned either by $\hat{U}^a$ or $\hat{v}_a$ eigenstates with eigenvalues $U^a \in \mathbb{R}$ or $v_a \in \mathbb{R}$, respectively,

$$\hat{U}^a|U^a\rangle = U^a|U^a\rangle, \qquad \hat{v}_a|v_a\rangle = v_a|v_a\rangle. \tag{75}$$

Once again our convention for relative phases is such that

$$|U^a\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}U^a\hat{v}_a}|U^a = 0\rangle, \qquad |v_a\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}v_a\hat{U}^a}|v_a = 0\rangle, \tag{76}$$

and hence

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}U^a_0\hat{v}_a}|U^a\rangle = \left|U^a + U^a_0\right\rangle, \qquad \mathrm{e}^{\frac{\mathrm{i}}{\hbar}v_{0a}\hat{U}^a}|v_a\rangle = |v_a + v_{0a}\rangle. \tag{77}$$

The noncommutative Weyl–Heisenberg group elements are parameterized according to

$$\begin{aligned} W(U^a, v_a; \varphi) &= \exp\left[\mathrm{i}\varphi\mathbb{I} + \frac{\mathrm{i}}{\hbar}v_a\hat{U}^a - \frac{\mathrm{i}}{\hbar}U^a\hat{v}_a\right] \\ &= \exp\left[\mathrm{i}\varphi\mathbb{I} + \frac{\mathrm{i}}{\hbar}v_a\hat{u}^a - \frac{\mathrm{i}}{\hbar}\left(u^a + \frac{\theta}{A\hbar}\epsilon^{ab}v_b\right)\hat{v}_a\right], \end{aligned} \tag{78}$$

where $u^a, U^a, v_a \in \mathbb{R}$ with the relations

$$U^a = u^a + \frac{\theta}{2A\hbar}\epsilon^{ab}v_b, \qquad u^a = U^a - \frac{\theta}{2A\hbar}\epsilon^{ab}v_b. \tag{79}$$

These operators are such that

$$\begin{aligned} W^\dagger(U^a, v_a; \varphi)\hat{u}^a W(U^a, v_a; \varphi) &= \hat{u}^a + u^a\mathbb{I}, \\ W^\dagger(U^a, v_a; \varphi)\hat{U}^a W(U^a, v_a; \varphi) &= \hat{U}^a + U^a\mathbb{I}, \\ W^\dagger(U^a, v_a; \varphi)\hat{v}_a W(U^a, v_a; \varphi) &= \hat{v}_a + v_a\mathbb{I}, \end{aligned} \tag{80}$$

while their group composition law is

$$W\left(U^a_2, v_{2a}; \varphi_2\right)W\left(U^a_1, v_{1a}; \varphi_1\right) = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(v_{2a}U^a_1 - U^a_2 v_{1a})}W\left(U^a_2 + U^a_1, v_{2a} + v_{1a}; \varphi_2 + \varphi_1\right), \tag{81}$$

implying the cocycle property

$$W\left(U^a_1, v_{1a}; \varphi_1\right)W\left(U^a_2, v_{2a}; \varphi_2\right) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(v_{1a}U^a_2 - v_{2a}U^a_1)}W\left(U^a_2, v_{2a}; \varphi_2\right)W\left(U^a_1, v_{1a}; \varphi_1\right). \tag{82}$$

For the reasons mentioned in the introduction, one may consider as translation operators $\hat{T}_i$ some arbitrary linear combination of $\hat{p}_i$ and $\epsilon_{ij}\hat{x}^j$, which both effect translations in the coordinate operators $\hat{x}^i$. Specifically, when imposing also the condition (12), the choice to be made is

$$\hat{T}_i = \left(1 - \frac{\beta\theta}{\hbar}\right) \hat{p}_i + \beta\epsilon_{ij}\hat{x}^j, \tag{83}$$

where $\beta \in \mathbb{R}$ is an arbitrary real variable, with appropriate physical dimension, parameterizing the freedom in the choice of translation operators. Note that even in the commutative case, $\theta = 0$, a nonvanishing β deforms the choice of translation group compared to the usual choice $\hat{T}_i = \hat{p}_i$, corresponding to $\beta = 0$. When $\theta \neq 0$, the value $\beta = \hbar/\theta$ corresponds to a choice of translation operators which is that of the ordinary noncommutative torus of section 3.

For later analysis, it is convenient to rather use the 'rectified' translation operators

$$\hat{T}_a = e^i_a \hat{T}_i = \left(1 - \frac{\beta\theta}{\hbar}\right) \hat{v}_a + \beta A \epsilon_{ab} \hat{u}^b = \left(1 - \frac{\beta\theta}{2\hbar}\right) \hat{v}_a + \beta A \epsilon_{ab} \hat{U}^b. \tag{84}$$

The relevant commutation relations are found to be

$$[\hat{u}^a, \hat{T}_b] = \mathrm{i}\hbar\delta^a_b\mathbb{I}, \qquad [\hat{U}^a, \hat{T}_b] = \mathrm{i}\hbar\left(1 - \frac{\beta\theta}{2\hbar}\right)\delta^a_b\mathbb{I}, \qquad [\hat{v}_a, \hat{T}_b] = \mathrm{i}\hbar\beta A\epsilon_{ab}\mathbb{I}, \tag{85}$$

while the algebra of the translation group is

$$[\hat{T}_a, \hat{T}_b] = \mathrm{i}\hbar 2\beta A\left(1 - \frac{\beta\theta}{2\hbar}\right)\epsilon_{ab}\mathbb{I}. \tag{86}$$

In view of the expression for $\hat{T}_a$, it proves useful to also introduce the operators

$$\hat{Q}_a = \left(1 - \frac{\beta\theta}{2\hbar}\right) \hat{v}_a - \beta A \epsilon_{ab} \hat{U}^b = \hat{v}_a - \beta A \epsilon_{ab} \hat{u}^b, \tag{87}$$

which are such that

$$\begin{aligned} &[\hat{u}^a, \hat{Q}_b] = \mathrm{i}\hbar\left(1 - \frac{\beta\theta}{\hbar}\right)\delta^a_b\mathbb{I}, \qquad [\hat{U}^a, \hat{Q}_b] = \mathrm{i}\hbar\left(1 - \frac{\beta\theta}{2\hbar}\right)\delta^a_b\mathbb{I}, \\ &[\hat{v}_a, \hat{Q}_b] = -\mathrm{i}\hbar\beta A\epsilon_{ab}\mathbb{I}, \end{aligned} \tag{88}$$

and

$$[\hat{Q}_a, \hat{Q}_b] = -\mathrm{i}\hbar 2\beta A\left(1 - \frac{\beta\theta}{2\hbar}\right)\epsilon_{ab}\mathbb{I}. \tag{89}$$

From this follows the important result

$$[\hat{T}_a, \hat{Q}_b] = 0. \tag{90}$$

However, since

$$\hat{Q}_a + \hat{T}_a = 2\left(1 - \frac{\beta\theta}{2\hbar}\right)\hat{v}_a, \qquad \hat{Q}_a - \hat{T}_a = -2\beta A\epsilon_{ab}\hat{U}^b, \tag{91}$$

it is only when $2\beta A(1 - \beta\theta/(2\hbar)) \neq 0$ that the algebra $(\hat{Q}_a, \hat{T}_a, \mathbb{I})$ is equivalent to any of the equivalent algebras $(\hat{x}^i, \hat{p}_i, \mathbb{I})$, $(\hat{u}^a, \hat{v}_a, \mathbb{I})$ or $(\hat{U}^a, \hat{v}_a, \mathbb{I})$. Under this condition one has the inverse relations

$$\hat{U}^a = \frac{1}{\beta A}\frac{1}{2}\epsilon^{ab}[\hat{Q}_b - \hat{T}_b], \qquad \hat{v}_a = \frac{1}{\left(1 - \frac{\beta\theta}{2\hbar}\right)}\frac{1}{2}[\hat{Q}_a + \hat{T}_a]. \tag{92}$$

Finally, under the same condition, $2\beta A(1 - \beta\theta/(2\hbar)) \neq 0$, the following expression is also of use when considering the NC-WH group elements introduced previously,

$$v_a\hat{U}^a - U^a\hat{v}_a = \frac{1}{2\beta A\left(1 - \frac{\beta\theta}{2\hbar}\right)}[Q_a\epsilon^{ab}\hat{Q}_b - T_a\epsilon^{ab}\hat{T}_b], \tag{93}$$

where

$$T_a = \left(1 - \frac{\beta\theta}{2\hbar}\right) v_a + \beta A \epsilon_{ab} U^b, \tag{94}$$

$$Q_a = \left(1 - \frac{\beta\theta}{2\hbar}\right) v_a - \beta A \epsilon_{ab} U^b. \tag{95}$$

In addition to the adjoint actions in (80), one also finds

$$\begin{aligned} W^\dagger(U^a, v_a; \varphi)\hat{T}_a W(U^a, v_a; \varphi) &= \hat{T}_a + T_a \mathbb{I}, \\ W^\dagger(U^a, v_a; \varphi)\hat{Q}_a W(U^a, v_a; \varphi) &= \hat{Q}_a + Q_a \mathbb{I}. \end{aligned} \tag{96}$$

Turning to the translation group elements

$$U(n^a) = C(n^a)\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} n^a \hat{T}_a}, \tag{97}$$

the Abelian composition law condition (10) implies the cocycle condition

$$\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} 2\beta A (1 - \frac{\beta\theta}{2\hbar}) \epsilon_{ab} n^a \ell^b} C(n^a) C(\ell^a) = C(n^a + \ell^a). \tag{98}$$

The general solution is of the form

$$C(n^a) = \mathrm{e}^{-\mathrm{i}\pi k_0 n^1 n^2}\, \mathrm{e}^{2\mathrm{i}\pi n^a \epsilon_{ab} \lambda^b}, \tag{99}$$

where $\lambda^a \in [0, 1[$ (modulo the integers) are, once again, $U(1)$ holonomy parameters, while $k_0 \in \mathbb{Z}$ is an integer such that

$$2\beta A \left(1 - \frac{\beta\theta}{2\hbar}\right) = 2\pi\hbar k_0, \qquad k_0 \in \mathbb{Z}, \qquad \beta \in \mathbb{R}. \tag{100}$$

This condition generalizes the area quantization condition (52), which applies to the ordinary noncommutative torus discussed in section 3, to the noncommutative Heisenberg algebra in the presence of the β parameter. In particular, for the choice $\beta = \hbar/\theta$, the integer k_0 must again be such that $A = 2\pi\theta k_0$.

As a function of A, θ and k_0, the allowed values for β are thus

$$\beta = \frac{\hbar}{\theta}\left[1 \pm \sqrt{1 - \frac{2\pi\theta}{A} k_0}\right], \qquad k_0 \leqslant \frac{A}{2\pi\theta}, \qquad k_0 \in \mathbb{Z}. \tag{101}$$

The choice $\beta = \hbar/\theta$ corresponds precisely to the degenerate case $A = 2\pi\theta k_0$ with $k_0 > 0$. The value $k_0 = 0$ is associated with the two distinct situations $\beta = 0$ or $\beta = 2\hbar/\theta$, namely $2\beta A(1 - \beta\theta/(2\hbar)) = 0$. This is also the situation when the translation generators $\hat{T}_a$ commute. For any fixed positive $k_0 > 0$, as the area A increases continuously from the minimal value $2\pi\theta k_0$, the two above branches of β values either decrease or increase from $\beta = \hbar/\theta$ towards the two singular values $\beta = 0$ or $\beta = 2\hbar/\theta$, respectively. Hence the interval $\beta \in]0, 2\hbar/\theta[$ is certainly distinguished when $k_0 \neq 0$ for any finite area A, while for a finite area A the two end points of that interval correspond only to the case with $k_0 = 0$. Strictly negative values of k_0 correspond to β values outside the interval $[0, 2\hbar/\theta]$. Note that in the commutative case the only surviving branch is such that

$$\theta = 0: \qquad \beta = \frac{\pi\hbar}{A} k_0, \qquad k_0 \in \mathbb{Z}. \tag{102}$$

Thus, besides the ordinary choice $\beta = 0$ corresponding to $k_0 = 0$, there still exist many other possibilities for a choice of translation operators. Of course, it is only when $\beta = 0$ that the momentum operators $\hat{p}_i$ are not affected by translations in configuration space.

In conclusion, the lattice group defining the noncommutative two-torus geometry is generated by the following elements of the NC-WH group,

$$U(n^a) = \mathrm{e}^{-\mathrm{i}\pi k_0 n^1 n^2}\,\mathrm{e}^{2\mathrm{i}\pi n^a \epsilon_{ab}\lambda^b}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}n^a\hat{T}_a} = W\left(\left(1-\frac{\beta\theta}{2\hbar}\right)n^a, \beta A\epsilon_{ab}n^b; 2\pi n^a\epsilon_{ab}\lambda^b - \pi k_0 n^1 n^2\right). \tag{103}$$

In particular, the translation shifts induced for each of the operators of interest, $U^\dagger(n^a)\hat{\mathcal{O}}U(n^a) = \hat{\mathcal{O}} + \Delta_n\mathcal{O}\mathbb{I}$, are such that

$$\begin{aligned}
\hat{\mathcal{O}} &= \hat{u}^a: && \Delta_n u^a = n^a,\\
\hat{\mathcal{O}} &= \hat{U}^a: && \Delta_n U^a = \left(1-\frac{\beta\theta}{2\hbar}\right)n^a,\\
\hat{\mathcal{O}} &= \hat{v}_a: && \Delta_n v_a = \beta A\epsilon_{ab}n^b,\\
\hat{\mathcal{O}} &= \hat{T}_a: && \Delta_n T_a = 2\beta A\left(1-\frac{\beta\theta}{2\hbar}\right)\epsilon_{ab}n^b = 2\pi\hbar k_0\epsilon_{ab}n^b,\\
\hat{\mathcal{O}} &= \hat{Q}_a: && \Delta_n Q_a = 0.
\end{aligned} \tag{104}$$

In order to proceed now with the construction of representations of the NC-2T-WH group, one needs to consider separately the distinct cases $k_0 = 0$ from the generic situation with $k_0 \neq 0$.

5. The distinct representations with $k_0 = 0$

5.1. The point $\beta = 0$

The degenerate case $\beta = 0$ corresponds to the choices

$$\hat{T}_a = \hat{v}_a, \qquad \hat{Q}_a = \hat{v}_a, \qquad k_0 = 0. \tag{105}$$

The lattice group then consists of the commuting elements

$$U(n^a) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}n^a(\hat{v}_a - 2\pi\hbar\lambda_a)} = W(n^a, 0; 2\pi n^a\lambda_a), \qquad \lambda_a = \epsilon_{ab}\lambda^b. \tag{106}$$

Consequently, the situation is comparable to the discussion in section 2 for the commuting Weyl–Heisenberg group. In particular, whether by considering the projection operator (13) or the above expression, it is clear that the subspace of invariant states is spanned by the following discrete set of $\hat{v}_a$ eigenstates,

$$|\overline{m}_a\rangle \equiv |\overline{v}_a\rangle, \qquad \overline{v}_a = 2\pi\hbar\left(\overline{m}_a + \lambda_a\right), \qquad \overline{m}_a \in \mathbb{Z}. \tag{107}$$

Considering now the NC-WH group elements $W(U^a, v_a; \varphi)$, based on the composition law (81), it readily follows that the invariance condition (18) implies the restriction

$$v_a = 2\pi\hbar m_a, \qquad m_a \in \mathbb{Z}. \tag{108}$$

Furthermore, for any such value of v_a, the invariance condition (19) leads to the following choice for the group parameter φ,

$$\varphi(U^a, m_a) = \pi U^a\left(m_a + 2\lambda_a\right). \tag{109}$$

Note that under lattice shifts the parameters (U^a, m_a) transform according to

$$\Delta_n U^a = n^a, \qquad \Delta_n m_a = 0. \tag{110}$$

Consequently, in the case $\beta = 0$ the two-torus noncommutative Weyl–Heisenberg group consists of all the following elements

$$W_0(U^a, m_a) = W(U^a, 2\pi\hbar m_a; \pi U^a(m_a + 2\lambda_a)) = \mathrm{e}^{2\mathrm{i}\pi m_a\hat{U}^a}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}U^a(\hat{v}_a - 2\pi\hbar\lambda_a)}, \tag{111}$$

where $U^a \in [0, 1[$ (modulo the integers) and $m_a \in \mathbb{Z}$. The representation of the group on the space of invariant states is

$$W_0(U^a, m_a)|\overline{m}_a\rangle = \mathrm{e}^{-2\mathrm{i}\pi U^a \overline{m}_a}|\overline{m}_a + m_a\rangle, \tag{112}$$

which is indeed single-valued under lattice shifts of the group parameters. Finally, the group composition law is

$$W_0\big(U_2^a, m_{2a}\big)W_0\big(U_1^a, m_{1a}\big) = \mathrm{e}^{-2\mathrm{i}\pi m_{1a}U_2^a}\,W_0\big(U_2^a + U_1^a, m_{2a} + m_{1a}\big), \tag{113}$$

which leads to the cocycle property

$$W_0\big(U_1^a, m_{1a}\big)W_0\big(U_2^a, m_{2a}\big) = \mathrm{e}^{2\mathrm{i}\pi(U_2^a m_{1a} - U_1^a m_{2a})}\,W_0\big(U_2^a, m_{2a}\big)W_0\big(U_1^a, m_{1a}\big). \tag{114}$$

In conclusion in the case $\beta = 0$, the representation of the noncommutative two-torus Weyl–Heisenberg algebra is discrete infinite dimensional, and essentially coincides with the representation of the Weyl–Heisenberg group on the commutative torus discussed in section 2.

5.2. The point $\beta = 2\hbar/\theta$

The value $\beta = 2\hbar/\theta$ corresponds to the second branch with $k_0 = 0$ and applies only in the noncommutative case, $\theta \neq 0$. This situation corresponds to the choice

$$\hat{T}_a = \frac{2A\hbar}{\theta}\epsilon_{ab}\hat{U}^b, \qquad \hat{Q}_a = -\hat{T}_a, \tag{115}$$

with the commutative translation algebra

$$[\hat{T}_a, \hat{T}_b] = 0. \tag{116}$$

The lattice group thus consists of the commuting elements

$$U(n^a) = \mathrm{e}^{-2\mathrm{i}\frac{A}{\theta}n^a\epsilon_{ab}(\hat{U}^b - \pi\frac{\theta}{A}\lambda^b)} = W\left(0, \frac{2A\hbar}{\theta}\epsilon_{ab}n^b; 2\pi n^a\epsilon_{ab}\lambda^b\right), \tag{117}$$

which induce the following lattice shifts

$$\Delta_n u^a = n^a, \qquad \Delta_n U^a = 0, \qquad \Delta_n v_a = \frac{2A\hbar}{\theta}\epsilon_{ab}n^b, \qquad \Delta_n T_a = 0, \qquad \Delta_n Q_a = 0. \tag{118}$$

From the above expression, or by considering the action of the projection operator (13), invariant states are seen to be spanned by the following discrete set of $\hat{U}^a$ eigenstates,

$$|\overline{k}^a\rangle \equiv |\overline{U}^a\rangle: \qquad \overline{U}^a = \frac{\pi\theta}{A}(\overline{k}^a + \lambda^a), \qquad \overline{k}^a \in \mathbb{Z}. \tag{119}$$

Considering now the invariance condition (18), based on the composition law (81), the following restriction arises for the parameters of the NC-WH group elements $W(U^a, v_a; \varphi)$,

$$U^a = \frac{\pi\theta}{A}k^a, \qquad k^a \in \mathbb{Z}. \tag{120}$$

Furthermore, given such a value for U^a, the requirement (19) leads to the following choice of parameter φ for those NC-WH transformations,

$$W(U^a, v_a; \varphi): \qquad \varphi(k^a, v_a) = -\frac{\pi\theta}{2A\hbar}\left(k^a + 2\lambda^a\right)v_a. \tag{121}$$

Note that under lattice shifts the parameters (k^a, v_a) transform according to

$$\Delta_a k^a = 0, \qquad \Delta_n v_a = \frac{2A\hbar}{\theta}\epsilon_{ab}n^b. \tag{122}$$

Consequently, in the case $\beta = 2\hbar/\theta$, the two-torus noncommutative Weyl–Heisenberg group consists of all the elements

$$W_0(k^a, v_a) = W\left(\frac{\pi\theta}{A}k^a, v_a; -\frac{\pi\theta}{2A\hbar}(k^a + 2\lambda^a)v_a\right) = \mathrm{e}^{-\mathrm{i}\frac{\pi\theta}{A\hbar}k^a\hat{v}_a}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\hat{U}^a - \frac{\pi\theta}{A}\lambda^a)}, \tag{123}$$

where $v_a \in [0, 2A\hbar/\theta[$ (modulo $2A\hbar/\theta$) and $k^a \in \mathbb{Z}$. The action of the group on the invariant states is

$$W_0(k^a, v_a)|\overline{k}^a\rangle = \mathrm{e}^{\mathrm{i}\frac{\pi\theta}{A\hbar}v_a\overline{k}^a}|\overline{k}^a + k^a\rangle, \tag{124}$$

which is indeed single-valued in lattice shifts of the group parameters (k^a, v_a). The group composition law is

$$W_0(k_2^a, v_{2a})W_0(k_1^a, v_{1a}) = \mathrm{e}^{\mathrm{i}\frac{\pi\theta}{A\hbar}v_{2a}k_1^a}W_0(k_2^a + k_1^a, v_{2a} + v_{1a}), \tag{125}$$

from which follows the cocycle property

$$W_0(k_1^a, v_{1a})W_0(k_2^a, v_{2a}) = \mathrm{e}^{\mathrm{i}\frac{\pi\theta}{A\hbar}(v_{1a}k_2^a - v_{2a}k_1^a)}W_0(k_2^a, v_{2a})W_0(k_1^a, v_{1a}). \tag{126}$$

In conclusion, in the case $\beta = 2\hbar/\theta$, the noncommutative two-torus Weyl–Heisenberg group possesses a single discrete infinite dimensional representation, very similar to the one for $\beta = 0$, except that in this case it is in the dual eigenspace of the $\hat{U}^a$ operators.

6. The generic representations with $k_0 \neq 0$

When $k_0 \neq 0$ the lattice group elements are given in (103). A basis of invariant states may be constructed in either the $\hat{U}^a$ or $\hat{v}_a$ eigensectors. In the latter case, let us introduce the notation

$$|\overline{v}^a\rangle \equiv |\overline{v}_a\rangle: \qquad \overline{v}_a = \frac{\beta A}{k_0}\epsilon_{ab}(\overline{v}^b + \lambda^b). \tag{127}$$

Considering either the projection operator (13) or the action of the lattice group on the states $|v_a\rangle$, it is found that invariant states are spanned by the combinations

$$|\overline{v}^a\rangle\rangle = \sum_{\ell^a\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi k_0\ell^1\ell^2 + \mathrm{i}\pi\ell^a\epsilon_{ab}\lambda^b - \mathrm{i}\pi n\ell^a\epsilon_{ab}\overline{v}^b}|\overline{v}^a + k_0\ell^a\rangle, \tag{128}$$

which possess, for $n^a \in \mathbb{Z}$, the following property:

$$|\overline{v}^a + k_0 n^a\rangle\rangle = \mathrm{e}^{\mathrm{i}\pi k_0 n^1 n^2 - \mathrm{i}\pi n^a\epsilon_{ab}\lambda^b + \mathrm{i}\pi n^a\epsilon_{ab}\overline{v}^b}|\overline{v}^a\rangle\rangle. \tag{129}$$

This shows that the two parameters $\overline{v}^a$ are indeed each defined modulo k_0.

Likewise in the $\hat{U}^a$ eigensector, let us introduce the notation

$$|\overline{\mu}^a\rangle \equiv |\overline{U}^a\rangle: \qquad \overline{U}^a = \frac{1}{k_0}\left(1 - \frac{\beta\theta}{2\hbar}\right)(\overline{\mu}^a + \lambda^a). \tag{130}$$

It is then found that invariant states are spanned by the combinations

$$|\overline{\mu}^a\rangle\rangle = \sum_{\ell^a\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi k_0\ell^1\ell^1 + \mathrm{i}\pi\ell^a\epsilon_{ab}\lambda^b - \mathrm{i}\pi\ell^a\epsilon_{ab}\overline{\mu}^b}|\overline{\mu}^a + k_0\ell^a\rangle, \tag{131}$$

which possess, for $n^a \in \mathbb{Z}$, the properties

$$|\overline{\mu}^a + k_0 n^a\rangle\rangle = \mathrm{e}^{\mathrm{i}\pi k_0 n^1 n^2 - \mathrm{i}\pi n^a\epsilon_{ab}\lambda^b + \mathrm{i}\pi n^a\epsilon_{ab}\overline{\mu}^b}|\overline{\mu}^a\rangle\rangle, \tag{132}$$

showing that the two parameters $\overline{\mu}^a$ are indeed each defined modulo k_0.

Considering the general NC-WH operators $W(U^a, v_a; \varphi)$ and their group composition law (81), the invariance condition (18) imposes the restriction

$$T_a = \left(1 - \frac{\beta\theta}{2\hbar}\right)v_a + \beta A\epsilon_{ab}U^b = 2\pi\hbar\epsilon_{ab}k^b, \qquad k^a \in \mathbb{Z}, \tag{133}$$

whereas the linearly independent combination

$$Q_a = \left(1 - \frac{\beta\theta}{2\hbar}\right) v_a - \beta A \epsilon_{ab} U^b = 2\pi\hbar\epsilon_{ab}\rho^a, \qquad \rho^a \in \mathbb{R}, \tag{134}$$

is left arbitrary. Note that lattice shifts induce the following transformations for the variables (k^a, ρ^a),

$$\Delta_n k^a = k_0 n^a, \qquad \Delta_n \rho^a = 0. \tag{135}$$

Furthermore, when this restriction is met, the second invariance condition (19) leads to the following choice for the group parameter φ:

$$\varphi(k^a, \rho^a) = -\frac{\pi}{k_0} k^1 k^2 + \frac{2\pi}{k_0}\epsilon_{ab} k^a \lambda^b. \tag{136}$$

Consequently, the NC-WH group elements are given by

$$W_0(k^a, \rho^a) = W\left(U^a, v_a; -\frac{\pi}{k_0} k^1 k^2 + \frac{2\pi}{k_0}\epsilon_{ab} k^a \lambda^b\right), \tag{137}$$

where

$$U^a = \left(1 - \frac{\beta\theta}{2\hbar}\right)\frac{1}{k_0}(k^a - \rho^a), \qquad v_a = \frac{\beta A}{k_0}\epsilon_{ab}(k^a + \rho^a). \tag{138}$$

As a matter of fact one also has (see (93))

$$W_0(k^a, \rho^a) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{k_0}k^1 k^2 + 2\mathrm{i}\pi\epsilon_{ab}\frac{k^a}{k_0}\lambda^b}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{\rho^a}{k_0}\hat{Q}_a}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{k^a}{k_0}\hat{T}_a}, \tag{139}$$

where $k^a \in \mathbb{Z}$ modulo k_0 and $\rho^a \in \mathbb{R}$. The representation of the group is such that when acting on the invariant states one finds

$$W_0(k^a, \rho^a)|\overline{v}^a\rangle\rangle = \mathrm{e}^{-\mathrm{i}\frac{\pi}{k_0}k^1k^2 - \mathrm{i}\frac{\pi}{k_0}\epsilon_{ab}k^a(\overline{v}^b - \lambda^b) - \mathrm{i}\frac{\pi}{k_0}\epsilon_{ab}(\overline{v}^a + k^a + \lambda^a)\rho^b}|\overline{v}^a + k^a + \rho^a\rangle\rangle, \tag{140}$$

$$W_0(k^a, \rho^a)|\overline{\mu}^a\rangle\rangle = \mathrm{e}^{-\mathrm{i}\frac{\pi}{k_0}k^1k^2 - \mathrm{i}\frac{\pi}{k_o}\epsilon_{ab}k^a(\overline{\mu}^b - \lambda^b) + \mathrm{i}\frac{\pi}{k_0}\epsilon_{ab}(\overline{\mu}^a + k^a + \lambda^a)\rho^b}|\overline{\mu}^a + k^a - \rho^a\rangle\rangle. \tag{141}$$

These actions may indeed be seen to be singled-valued under lattice shifts of the group parameters[11] k^a.

Hence, in contradistinction to all other representations discussed so far, and in particular that of the ordinary noncommutative torus in the absence of the momentum operators, the generic irreducible representation of the noncommutative two-torus Weyl–Heisenberg group with $k_0 \neq 0$ is noncountable infinite dimensional and spanned by a collection of states labelled by two continuous parameters each defined modulo k_0.

It is clear that by identifying appropriate subsets of the group parameters (k^a, ρ^a), which are closed under addition, i.e., closed under composition within the NC-2T-WH group, subgroups may be identified for which the above representation space becomes reducible, possibly leading to discrete infinite dimensional representations of such subgroups, or even finite dimensional ones. For instance considering only those NC-2T-WH group elements with $\rho^a = 0$, the above representation space separates into an infinite noncountable ensemble of finite $|k_0|$-dimensional representations of that subgroup. As seen from (139), one then in fact constructs a representation of the subalgebra

$$[\hat{T}_a, \hat{T}_b] = \mathrm{i}\hbar 2\pi\hbar k_0 \epsilon_{ab}\mathbb{I} \tag{142}$$

[11] The composition law and cocycle properties are given hereafter. We leave aside the construction of an inner product on these representation spaces, as well as for those in the two distinguished cases with $k_0 = 0$. This is rather straightforward. Note that in the present case with $k_0 \neq 0$ the invariant states are not normalizable since they belong to a continuous set.

of the original full noncommutative Heisenberg algebra. Since this subalgebra is isomorphic to that of the ordinary noncommutative two-torus in section 3,

$$[\hat{u}^a, \hat{u}^b] = \frac{\mathrm{i}}{2\pi k_0}\epsilon^{ab}\mathbb{I}, \tag{143}$$

and as the torus topology is defined through these operators as translation operators, the irreducible representation of the pure $\hat{T}_a$ algebra must indeed again be of finite dimension $|k_0|$ for some integer k_0. Of course when $A = 2\pi\theta k_0$ and thus $\beta = \hbar/\theta$, such a reduction coincides precisely with the construction in section 3.

In a likewise manner more involved subgroups may be imagined in which even nonvanishing parameters ρ^a of rational values are used, but as was already remarked at the end of section 2 in the commutative case, the genuine NC-2T-WH group corresponds to all elements $W_0(k^a, \rho^a)$ for the entire ranges of allowed values for the group parameters (k^a, ρ^a). It is thus quite remarkable that by just extending the ordinary noncommutative configuration space algebra of operators $\hat{x}^i$ with the momentum operators $\hat{p}_i$ on a configuration space having the topology of a torus, the irreducible representation of finite dimension k_0 of the k_0^2-dimensional finite noncommutative Weyl–Heisenberg group of section 3 turns into a noncountable infinite dimensional representation labelled by two real variables, each defined modulo k_0, of a group which itself has become the semi-direct product of a finite k_0^2-dimensional group and a Lie group parameterized by the coordinates $\rho^a \in \mathbb{R}$ with the specific composition law and cocycle properties,

$$W_0\big(k_2^a, \rho_2^a\big)W_0\big(k_1^a, \rho_1^a\big) = \mathrm{e}^{\frac{2\mathrm{i}\pi}{k_0}k_1^1k_2^2 + \frac{\mathrm{i}\pi}{k_0}\epsilon_{ab}\rho_2^a\rho_1^b}\, W_0\big(k_2^a + k_1^a, \rho_2^a + \rho_1^a\big), \tag{144}$$

$$W_0\big(k_1^a, \rho_1^a\big)W_0\big(k_2^a, \rho_2^2\big) = \mathrm{e}^{\frac{2\mathrm{i}\pi}{k_0}\epsilon_{ab}(\rho_1^a\rho_2^b - k_1^ak_2^b)}\, W_0\big(k_2^a, \rho_2^a\big)W_0\big(k_1^a, \rho_1^a\big). \tag{145}$$

7. The free particle and its energy spectrum

Given the considerations discussed in the introduction, the choice of the Hamiltonian operator for the description of the (nonrelativistic) free particle's motion on the noncommutative torus should be of the form

$$\hat{H} = \tfrac{1}{2}h_0\delta^{ij}\hat{\Pi}_i\hat{\Pi}_j, \qquad h_0 > 0, \quad h_0 \in \mathbb{R}, \tag{146}$$

where $\hat{\Pi}_i$ are operators built out of linear combinations of $\hat{x}^i$ and $\hat{p}_i$ which ought to commute with the choice of translation operators $\hat{T}_i$ in terms of which the torus lattice group is constructed. This issue and the ensuing energy spectrum will now be considered for each of the classes of representations addressed in the previous sections.

7.1. The ordinary general torus

In the ordinary commutative case with the choice of translation operators $\hat{T}_i = \hat{p}_i$, the operators $\hat{\Pi}_i$ that commute with these are clearly the momentum operators themselves, $\hat{\Pi}_i = \hat{p}_i$. Consequently,

$$\hat{H} = \frac{1}{2}h_0\delta^{ij}\hat{p}_i\hat{p}_j, \qquad h_0 = \frac{1}{\mu}. \tag{147}$$

Since the space of invariant states is spanned by the momentum eigenstates

$$|\overline{m}_a\rangle, \qquad \overline{p}_i = 2\pi\hbar\tilde{e}_i^a(\overline{m}_a + \lambda_a), \qquad \overline{m}_a \in \mathbb{Z}, \tag{148}$$

the eigenstates of the Hamiltonian consist precisely of these invariant states with the energy eigenspectrum

$$E(\overline{m}_a) = \tfrac{1}{2}(2\pi\hbar)^2h_0g^{ab}(\overline{m}_a + \lambda_a)(\overline{m}_b + \lambda_b). \tag{149}$$

7.2. The ordinary noncommutative torus

In the case of the ordinary noncommutative algebra (22), it may readily be established that any operator that is quadratic in the basic coordinate operators $\hat{x}^i$, and which commutes with the translation operators $\hat{T}_i$, which are in effect again the $\hat{x}^i$, is necessarily proportional to the unit operator, $\mathbb{I}$. Consequently, in this situation the spectrum of the free noncommutative particle is degenerate for each of its k_0 independent states for a torus area quantized in units of θ, $A = 2\pi\theta k_0$.

This conclusion is in accord with the fact that this specific situation is reached as the lowest Landau level projection of the ordinary Landau problem in the absence of any other interaction besides the coupling to the external homogeneous magnetic field. All such states are indeed degenerate and of finite number for a torus topology of quantized area [6].

7.3. The distinct representations with $k_0 = 0$

For the complete noncommutative Heisenberg algebra for which the translation operators are chosen to be the quantities $\hat{T}_a$, defined in terms of the parameter β, we know that the operators $\hat{Q}_a$ commute with $\hat{T}_a$, so that the general choice of the Hamiltonian is

$$\hat{H} = \tfrac{1}{2}h_0 g^{ab}\hat{Q}_a\hat{Q}_b = \tfrac{1}{2}h_0\delta^{ij}\hat{\Pi}_i\hat{\Pi}_j, \tag{150}$$

with

$$\hat{\Pi}_i = \tilde{e}_i^a\hat{Q}_a, \qquad \hat{Q}_a = e_a^i\hat{\Pi}_i. \tag{151}$$

When the choice $\beta = 0$ is made, corresponding to $k_0 = 0$ with

$$\hat{T}_a = \hat{v}_a = \hat{Q}_a, \qquad \hat{\Pi}_i = \hat{p}_i, \tag{152}$$

the space of invariant states is spanned by the $\hat{v}_a$ eigenstates

$$|\overline{m}_a\rangle: \qquad \overline{v}_a = 2\pi\hbar(\overline{m}_a + \lambda_a), \qquad \overline{m}_a \in \mathbb{Z}. \tag{153}$$

Consequently, these states are also the energy eigenstates with energy spectrum

$$E(\overline{m}_a) = \tfrac{1}{2}(2\pi\hbar)^2 h_0 g^{ab}(\overline{m}_a + \lambda_a)(\overline{m}_b + \lambda_b). \tag{154}$$

Hence this spectrum is independent of the noncommutativity parameter θ and in fact coincides with the one for the commutative particle.

Likewise, when the choice $\beta = 2\hbar/\theta$ is made, corresponding to $k_0 = 0$ with

$$\hat{T}_a = \frac{2A\hbar}{\theta}\epsilon_{ab}\hat{U}^b = -\hat{Q}_a, \qquad \hat{\Pi}_i = \hat{p}_i - \frac{2\hbar}{\theta}\epsilon_{ij}\hat{x}^j = -\hat{T}_i, \tag{155}$$

the space of invariant states is spanned by the $\hat{U}^a$ eigenstates

$$|\overline{k}^a\rangle: \qquad \overline{U}^a = \frac{\pi\theta}{A}(\overline{k}^a + \lambda^a), \qquad \overline{k}^a \in \mathbb{Z}. \tag{156}$$

These are thus also the energy eigenstates of the free particle for that choice of representation, with the energy spectrum

$$E(\overline{k}^a) = \tfrac{1}{2}(2\pi\hbar)^2 h_0 g^{ab}(\overline{k}^a + \lambda^a)(\overline{k}^b + \lambda^b). \tag{157}$$

Again this spectrum is independent of θ and coincides with the case when either $\beta = 0$ or $\theta = 0$.

7.4. The generic representations with $k_0 \neq 0$

In the generic situation with $k_0 \neq 0$, given that the Hamiltonian is of the form (150), the relevant operators $\hat{\Pi}_i$ are

$$\hat{\Pi}_i = \tilde{e}_i^a \hat{Q}_a = \left(1 - \frac{\beta\theta}{2\hbar}\right)\hat{p}_i - \beta\epsilon_{ij}\hat{X}^j = \hat{p}_i - \beta\epsilon_{ij}\hat{x}^j, \tag{158}$$

while the translation operators are

$$\hat{T}_a = e_a^i\left[\left(1 - \frac{\beta\theta}{2\hbar}\right)\hat{p}_i + \beta\epsilon_{ij}\hat{X}^j\right] = e_a^i\left[\left(1 - \frac{\beta\theta}{\hbar}\right)\hat{p}_i + \beta\epsilon_{ij}\hat{x}^j\right]. \tag{159}$$

It then proves useful to introduce the following Fock algebra of operators,

$$\begin{aligned} A_i &= \sqrt{\frac{A}{2\pi\hbar^2 k_0}}\left[\beta\hat{X}^i + \mathrm{i}\left(1 - \frac{\beta\theta}{2\hbar}\right)\hat{p}_i\right], \\ A_i^\dagger &= \sqrt{\frac{A}{2\pi\hbar^2 k_0}}\left[\beta\hat{X}^i - \mathrm{i}\left(1 - \frac{\beta\theta}{2\hbar}\right)\hat{p}_i\right], \end{aligned} \tag{160}$$

as well as

$$A_\pm = \frac{1}{\sqrt{2}}[A_1 \mp \mathrm{i}A_2], \qquad A_\pm^\dagger = \frac{1}{\sqrt{2}}\left[A_1^\dagger \pm \mathrm{i}A_2^\dagger\right], \tag{161}$$

such that

$$\left[A_i, A_j^\dagger\right] = \delta_{ij}\mathbb{I}, \qquad \left[A_\pm, A_\pm^\dagger\right] = \mathbb{I}. \tag{162}$$

Inverting these relations, and upon substitution into the appropriate expressions, one finds

$$\hat{H} = 2\pi\hbar^2 k_0 \frac{h_0}{A}\left[A_+^\dagger A_+ + \frac{1}{2}\right], \tag{163}$$

as well as

$$U(n^a) = \mathrm{e}^{-\mathrm{i}\pi k_0 n^1 n^2 + 2\mathrm{i}\pi n^a \epsilon_{ab}\lambda^b}\, \mathrm{e}^{\sqrt{\frac{\pi k_0}{A}}\left(n^a e_a^+ A_-^\dagger - n^a e_a^- A_-\right)}, \tag{164}$$

where $e_a^\pm = e_a^1 \pm \mathrm{i}e_a^2$.

Considering first the NC-H algebra on the noncommutative plane rather than the two-torus, the eigenstates of the Hamiltonian are given by

$$|k_+, k_-\rangle = \frac{1}{\sqrt{k_+!k_-!}}\left(A_+^\dagger\right)^{k_+}\left(A_-^\dagger\right)^{k_-}|0\rangle, \qquad k_+, k_- \in \mathbb{N}, \tag{165}$$

$|0\rangle$ being the Fock vacuum for the $\left(A_\pm, A_\pm^\dagger\right)$ Fock algebras, $(A_\pm|0\rangle = 0)$, and have eigenvalues

$$E(k_+, k_-) = 2\pi\hbar^2 k_0 \frac{h_0}{A}\left[k_+ + \frac{1}{2}\right]. \tag{166}$$

Note that this energy spectrum is once again independent of the noncommutativity parameter θ. Furthermore, it is infinitely degenerate in the excitations of the $\left(A_-, A_-^\dagger\right)$ sector, but possesses a harmonic finite gap in the excitations of the $\left(A_+, A_+^\dagger\right)$ sector, very much like the degenerate Landau problem on the plane, the rôle of the magnetic field being taken up here essentially by the integer $k_0 \neq 0$, or equivalently the parameter $\beta \neq 0, 2\hbar/\theta$ according to the quantization condition (100).

In order to identify now the energy eigenstates within the two-torus representation, it suffices to apply the projection operator (13) defined by the lattice group, since by construction in the free particle case the Hamiltonian operator commutes with the translation operators.

Consequently, the following projected energy eigenstates provide a basis for the two-torus representation:

$$|k_+, k_-\rangle\rangle = \mathbb{P}|k_+, k_-\rangle = \sum_{\ell^a \in \mathbb{Z}} U(\ell^a)|k_+, k_-\rangle. \tag{167}$$

Explicitly they read

$$|k_+, k_-\rangle\rangle = \frac{1}{\sqrt{k_+! k_-!}} \left(A_+^\dagger\right)^{k_+} \sum_{\ell^a \in \mathbb{Z}} \left[A_-^\dagger - \sqrt{\frac{\pi k_0}{A}} e_a^- \ell^a \mathbb{I} \right]^{k_-} U(\ell^a)|0\rangle, \tag{168}$$

and the energy spectrum is given in (166). Leaving aside the explicit construction of a new inner product on this subspace for which these invariant energy eigenstates would be orthonormalized[12], and the ensuing identification of the changes of bases $\langle\langle\overline{\mu}^a|k_+, k_-\rangle\rangle$ and $\langle\langle\overline{\nu}^a|k_+, k_-\rangle\rangle$, the important conclusion of the above analysis is that even upon compactification onto the two-torus geometry, irrespective of the choice of representation labelled by $k_0 \neq 0$, the spectrum of the free noncommutative particle remains totally independent of the noncommutativity parameter θ.

8. Conclusions

In order to identify possible observable consequences of noncommutative space coordinates in deformations of quantum mechanical systems, the present work considered the construction of the representations of the noncommutative Heisenberg algebra of the position and momentum operators, $\hat{x}^i$ and $\hat{p}_i$, when the configuration space topology and geometry is that of a flat two-torus. Allowing for a general definition of the torus topology through translations in the Euclidean configuration plane which may also transform the momentum spectrum, all possible representations have been identified. They fall into two classes, according to whether an integer k_0 labelling them is vanishing or not. When that integer k_0 vanishes, two distinct representations are possible, and are essentially isomorphic to the representations of the ordinary commutative Weyl–Heisenberg algebra on the torus spanned by a discrete spectrum of the quantized momentum eigenstates and labelled by $U(1)$ holonomy parameters. When the integer k_0 is nonvanishing, translations in configuration space also shift momentum eigenvalues, and representations of the Weyl–Heisenberg group are then continuous and spanned by eigenstates of the momentum operators, say, of which the spectrum belongs to the fundamental domain of some lattice structure related to the torus topology.

Note that when the configuration space translation operators are taken to be the coordinate operators themselves, as is the case for the usual discussion of the noncommutative torus which only considers the algebra of the position operators, a quantized torus area results. In contrast, by simply extending the algebra to also include the momentum operators, the representation space changes from finite dimensional to a noncountable infinite dimensional space spanned by points belonging to some fundamental domain.

In contrast with the single representation of the noncommutative Heisenberg algebra and Weyl–Heisenberg group on the Euclidean plane, which is also equivalent to the commutative representation, a rich structure of possible representations of the noncommutative Heisenberg algebra and Weyl–Heisenberg group results on the torus. Yet, despite this rich structure, when the dynamics of a free particle is considered, for whatever choice possible among the available representations, no physical consequence of noncommutativity is implied. Presumably this

[12] With respect to the inner product for the original orthonormalized Fock states $|k_+, k_-\rangle$, the invariant two-torus states $|k_+, k_-\rangle\rangle$ are not normalizable.

conclusion is unavoidable in the presence of a symmetry surviving the noncommutative deformation, namely translations in configuration space, as is also the situation for the free particle on the noncommutative plane.

Hence, as discussed already in the introduction, eventual observable effects of noncommutativity must be intertwined with effects from interactions, which makes it difficult to disentangle the role of noncommutativity and interactions on such fuzzy spaces since, at least in some approximations, interactions may effectively be represented through noncommutativity [4, 5]. The simplest manner in which to consider interactions and still move away as little as possible from a free particle dynamics is by confining the latter in a finite domain in configuration space through some (infinite) well potential, in effect introducing interactions only through boundary conditions. In the presence of noncommuting space coordinates this is not readily achieved and a dedicated approach needs to be developed. Work on this problem is being pursued and will be reported on elsewhere.

Acknowledgments

JG acknowledges the Institute of Theoretical Physics for an invited research staff position at the University of Stellenbosch (Republic of South Africa). He is most grateful to Professors Hendrik Geyer and Frederik Scholtz, and the School of Physics for their warm and generous hospitality during his sabbatical leave, and for financial support. His stay in South Africa is also supported in part by the Belgian National Fund for Scientific Research (FNRS) through a travel grant. He also acknowledges the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) Visiting Scholar Programme in support of a Visiting Professorship at the UNESCO-ICMPA (Republic of Benin). The work of JG is also supported by the Institut Interuniversitaire des Sciences Nucléaires and by the Belgian Federal Office for Scientific, Technical and Cultural Affairs through the Interuniversity Attraction Poles (IAP) P6/11.

Note added in proof. For what concerns the construction of representations of the noncommutative Weyl–Heisenberg group on the two-torus, some aspects of our analysis have overlap with some of the results in [8] (which have been extended to arbitrary Riemann surfaces in [9]). Both approaches however, are certainly different, and of complementary interest.

Appendix

With respect to a choice of the Cartesian coordinates x^i in the plane, the two-torus geometry is characterized by the lattice vectors e^i_a $(a, i = 1, 2)$ and their dual vectors $\tilde{e}^a_i$ such that

$$e^i_a \tilde{e}^b_i = \delta^b_a, \qquad \tilde{e}^a_i e^j_a = \delta^j_i. \tag{A.1}$$

The two-torus is thus defined by the equivalence relation

$$x^i \sim x^i + n^a e^i_a, \qquad n^a \in \mathbb{Z}. \tag{A.2}$$

The torus area is given by

$$A = \sqrt{\det g_{ab}}, \qquad g_{ab} = \delta_{ij} e^i_a e^j_b, \tag{A.3}$$

with the inverse metric

$$g^{ab} = \delta^{ij} \tilde{e}^a_i \tilde{e}^b_j, \qquad g_{ac} g^{cb} = \delta^b_a, \qquad g^{ac} g_{cb} = \delta^a_b. \tag{A.4}$$

The orientation of the two basis vectors $\left(e^i_1, e^i_2\right)$, in that order, is assumed to be such that

$$\det e^i_a > 0. \tag{A.5}$$

Then

$$A = \det e_a^i, \qquad \frac{1}{A} = \det \tilde{e}_i^a, \tag{A.6}$$

together with

$$\begin{aligned} \epsilon_{ij} e_a^i e_b^j &= A\epsilon_{ab}, & \epsilon^{ab} e_a^i e_b^j &= A\epsilon^{ij}, \\ \epsilon^{ij} \tilde{e}_i^a \tilde{e}_j^b &= \frac{1}{A}\epsilon^{ab}, & \epsilon_{ab} \tilde{e}_i^a \tilde{e}_j^b &= \frac{1}{A}\epsilon_{ij}, \end{aligned} \tag{A.7}$$

as well as

$$\begin{aligned} \epsilon_{ij} e_a^j &= A\tilde{e}_i^b \epsilon_{ba}, & \epsilon_{ab} \tilde{e}_i^b &= \frac{1}{A} e_a^j \epsilon_{ji}, \\ \epsilon^{ij} \tilde{e}_j^a &= \frac{1}{A} e_b^i \epsilon^{ba}, & \epsilon^{ab} e_b^i &= A\tilde{e}_j^a \epsilon^{ji}, \end{aligned} \tag{A.8}$$

where the antisymmetric symbols ϵ_{ab} and ϵ_{ij} are such that

$$\epsilon_{ij} = \epsilon^{ij}, \qquad \epsilon_{ab} = \epsilon^{ab}, \qquad \epsilon^{12} = +1 = \epsilon_{12}. \tag{A.9}$$

References

[1] Snyders H S 1947 *Phys. Rev.* **71** 38
[2] Rivasseau V 2007 *Non-commutative Renormalization* Poincaré Seminar, 82 pp (*Preprint* hep-th/0705.0705)
[3] Fiore G and Wess J 2007 *Phys. Rev.* D **75** 105022
[4] Scholtz F G, Chakraborty B, Gangopadhyay S and Govaerts J 2005 *J. Phys. A: Math. Gen.* **38** 9849
[5] Scholtz F G, Chakraborty B, Gangopadhyay S and Hazra A G 2005 *Phys. Rev.* D **71** 085005
[6] Connes A 1994 *Noncommutative Geometry* (London: Academic)
[7] Govaerts J and Villanueva V M 2000 *Int. J. Mod. Phys.* A **15** 4903
Govaerts J and Payen F 2007 *Mod. Phys. Lett.* A **22** 119
[8] Morariu B and Polychronakos A P 2001 *Nucl. Phys.* B **610** 531
[9] Morariu B and Polychronakos A P 2002 *Nucl. Phys.* B **634** 326